Problem 1) Suppose two burgers and a shake cost $8.00, and three burgers and two shakes cost $13.10. What is the cost of a burger? [Problem submitted by Vin Lee, LACC Professor of Mathematics, Source: Vin Lee]

Solution:

Let B be the cost of a burger and S the cost of a shake. $2B + S = \$8.00$ and $3B + 2S = \$13.10$. Use substitution or elimination to find that B = $2.90.

Problem 2) Find the minimum value of y if $y = x^2 - 4x + 3 + |x - 4|$ [Problem submitted by Anatoliy Nikolaychuk, LACC Professor of Mathematics, Source: Anatoliy Nikolaychuk]

Solution:

If $x \geq 4$, then $|x - 4| = x - 4$. So, $y = x^2 - 3x - 1$, whose graph is an upward opening parabola with a vertex $\left(\frac{3}{2}, -\frac{13}{4}\right)$. Since the function is increasing to the right of the vertex, the minimum y-value in this case occurs when x = 4 and is y = 3.

If $x < 4$, then $|x - 4| = 4 - x$. So, $y = x^2 - 5x + 7$, whose graph is an upward opening parabola with the minimum y-value at the vertex, $\left(\frac{5}{2}, \frac{3}{4}\right)$. The minimum y-value in this case is $\frac{3}{4}$.

Therefore, the answer to the question is $\frac{3}{4}$.

Problem 3) Find the least value of x such that $x>1$ and $\frac{1}{(\ln x)^{\ln x}} \le \frac{1}{x^2}$. [Problem submitted by Vin Lee, LACC Professor of Mathematics, Source: Vin Lee]

Solution:

$x^2 \le (\ln x)^{\ln x}$

$\ln x^2 \le \ln(\ln x)^{\ln x}$

$2\ln x \le (\ln x)\ln(\ln x)$

$2 \le \ln(\ln x)$

$e^2 \le \ln x$

$e^{(e^2)} \le x$

Problem 4) Find the maximum area of a rectangle inscribed in an equilateral triangle each of whose sides has a length of L. [Problem submitted by Vin Lee, LACC Professor of Mathematics, Source: Vin Lee]

Solution: Situate the triangle in the xy-plane with one of its sides on the x-axis and the opposite vertex on the positive y-axis. The line containing the side of the triangle in the first quadrant has a y-intercept $(0, \frac{\sqrt{3}}{2}L)$, an x-intercept $(\frac{1}{2}L, 0)$, a slope of $-\sqrt{3}$, and equation $y = -\sqrt{3}x + \frac{\sqrt{3}}{2}L$. Let (x, y) be the point on that line which is a vertex of the inscribed rectangle. Then 2x is the length of one side of the rectangle and y is the length of the adjacent side of the rectangle. So the area of the rectangle is

$$A = 2xy$$

$$A(x) = 2x(-\sqrt{3}x + \frac{\sqrt{3}}{2}L)$$

If we graph $A(x)$ using the vertical axis as the A-axis and the horizontal axis as the x-axis, the graph is a downward opening parabola whose vertex is $(\frac{1}{4}L, \frac{\sqrt{3}}{8}L^2)$. Therefore, the maximum area of an inscribed rectangle is $\frac{\sqrt{3}}{8}L^2$.

Problem 5) Find the range of the function $f(x)=\log_{\frac{1}{2}}\left(x-\frac{1}{2}\right)+\log_2\left(\sqrt{4x^2-4x+1}\right)$. [Problem submitted by Anatoliy Nikolaychuk, LACC Professor of Mathematics, Source: AMATYC]

Solution:

$$f(x)=\log_2\left(x-\frac{1}{2}\right)^{-1}+\log_2\left(\sqrt{(2x-1)^2}\right)$$

$$f(x)=-\log_2\left(x-\frac{1}{2}\right)+\log_2(2x-1)$$

$$f(x)=-\log_2\left(x-\frac{1}{2}\right)+\log_2\left(2\left(x-\frac{1}{2}\right)\right)$$

$$f(x)=-\log_2\left(x-\frac{1}{2}\right)+\log_2 2+\log_2\left(x-\frac{1}{2}\right)$$

$f(x)=1$ So, the range of f(x) is {1}.

Problem 6) Let k be a real number such that $\sqrt{x-3}+\sqrt{6-x}\geq k$ has a real solution. What is the maximum value of k? [Problem submitted by Angela Wayne, LACC Professor of Mathematics, Source: Angela Wayne]

Solution:

Let $y=\sqrt{x-3}+\sqrt{6-x}$, $3\leq x\leq 6$. Consider the equation
$y^2=(x-3)+2\sqrt{x-3}\sqrt{6-x}+(6-x)$

Note that $2\sqrt{x-3}\sqrt{6-x}=2\sqrt{-x^2+9x-18}$. Let $h(x)=-x^2+9x-18$, whose graph is a downward opening parabola with a maximum value of h occurring the vertex, which is

$h\left(\frac{9}{2}\right)=\frac{9}{4}$. So the maximum value of $2\sqrt{x-3}\sqrt{6-x}$ is $2\sqrt{\frac{9}{4}}=3$. Therefore,

$y^2=(x-3)+2\sqrt{x-3}\sqrt{6-x}+(6-x)\leq(x-3)+3+(6-x)=6$. So, the answer to the question is $\sqrt{6}$.

Problem 7) What is the minimum value of $f(x) = \max\{\sqrt{x-2}, |x-3|\}$, $x \geq 2$? [Problem submitted by Anatoliy Nikolaychuk, LACC Professor of Mathematics, Source: AMATYC]

Solution: Let $g(x) = \sqrt{x-2}$ and $h(x) = |x-3|$. From the graphs of $g(x)$ and $h(x)$ it can be seen that there exist two numbers, x_1 and x_2, such that $f(x_1) = g(x_1) = h(x_1)$ and $f(x_2) = g(x_2) = h(x_2)$. So, the minimum value of $f(x)$ is $f(x_1)$.

Let $g(x) = h(x)$

$$[g(x)]^2 = [h(x)]^2$$

$$\left(\sqrt{x-2}\right)^2 = \left(|x-3|\right)^2$$

$$x - 2 = x^2 - 6x + 9$$

$$0 = x^2 - 7x + 11$$

$$x = \frac{7 \pm \sqrt{5}}{2}$$

So, $x_1 = \frac{7-\sqrt{5}}{2}$ and $x_2 = \frac{7+\sqrt{5}}{2}$.

Substitute x_1 into either $g(x)$ or $h(x)$ to get $f(x_1) = \frac{\sqrt{5}-1}{2}$.

Problem 8) Let f(x) be a decreasing function whose domain is $(0,\infty)$. If $f(2a^2+a+1) < f(3a^2-4a+1)$, what are the possible values of a? [Problem submitted by Vin Lee, LACC Professor of Mathematics, Source: Mathematics Olympiad in China by Xiong Bin and Lee Peng Yee, East China Normal University Press, 2007]

Solution: The domain of f(x) implies $2a^2+a+1>0$ and $3a^2-4a+1>0$

$$2a^2+a+1 = 2\left(a+\frac{1}{4}\right)^2+\frac{7}{8}>0 \qquad \rightarrow \quad 0<a<\infty$$

$$3a^2-4a+1=(3a-1)(a-1)>0 \qquad \rightarrow \quad 0<a<\frac{1}{3} \text{ or } a>1 \; *$$

Also, since $f(x)$ is decreasing and $f(2a^2+a+1) < f(3a^2-4a+1)$

$$3a^2-4a+1<2a^2+a+1 \qquad \rightarrow \qquad a(a-5)<0 \qquad \rightarrow \qquad 0<a<5 \; **$$

Now combine * and ** to get $0<a<\frac{1}{3}$ or $1<a<5$.

Problem 9) Solve for x: $\sqrt[3]{1+\sqrt{x}}+\sqrt[3]{1-\sqrt{x}}=\sqrt[3]{5}$ [Problem submitted by Roger Wolf, LACC Professor of Mathematics, Source: Roger Wolf]

Solution: Cube both sides to get

$$1+\sqrt{x}+3\left(\sqrt[3]{1+\sqrt{x}}\right)^2\left(\sqrt[3]{1-\sqrt{x}}\right)+3\left(\sqrt[3]{1+\sqrt{x}}\right)\left(\sqrt[3]{1-\sqrt{x}}\right)^2+1-\sqrt{x}=5$$

$$2+3\left(\sqrt[3]{1+\sqrt{x}}\right)^2\left(\sqrt[3]{1-\sqrt{x}}\right)+3\left(\sqrt[3]{1+\sqrt{x}}\right)\left(\sqrt[3]{1-\sqrt{x}}\right)^2=5$$

$$3\left(\sqrt[3]{1+\sqrt{x}}\right)^2\left(\sqrt[3]{1-\sqrt{x}}\right)+3\left(\sqrt[3]{1+\sqrt{x}}\right)\left(\sqrt[3]{1-\sqrt{x}}\right)^2=3$$

$$\left(\sqrt[3]{1+\sqrt{x}}\right)^2\left(\sqrt[3]{1-\sqrt{x}}\right)+\left(\sqrt[3]{1+\sqrt{x}}\right)\left(\sqrt[3]{1-\sqrt{x}}\right)^2=1$$

$$\left(\sqrt[3]{1-\sqrt{x}}\right)\left(\sqrt[3]{1+\sqrt{x}}\right)\left(\sqrt[3]{1+\sqrt{x}}+\sqrt[3]{1-\sqrt{x}}\right)=1$$

$\left(\sqrt[3]{1-x}\right)\left(\sqrt[3]{1+\sqrt{x}}+\sqrt[3]{1-\sqrt{x}}\right)=1$, Next substitute $\sqrt[3]{5}$ for $\sqrt[3]{1+\sqrt{x}}+\sqrt[3]{1-\sqrt{x}}$ (from the original equation) to get

$$\left(\sqrt[3]{1-x}\right)\left(\sqrt[3]{5}\right)=1$$

$$5(1-x)=1$$

$$x=\frac{4}{5}$$

Problem 10) Find all solutions (real and/or complex) to $x^4 + 1 = 2x(x^2 + 1)$ [Problem submitted by Iris Magee and Roger wolf, LACC Professors of Mathematics, Source: Iris Magee]

Solution:

$x^4 + 1 - 2x(x^2 + 1) = 0$

$x^4 - 2x^3 - 2x + 1 = 0$

Let $f(x) = x^4 - 2x^3 - 2x + 1$

Using Descartes' Rule of Signs, there are two changes in sign of f(x) implying there are either 0 or 2 positive real roots, and there are zero changes in sign of f(-x) implying there are no negative real roots.

$f(1) = -2 < 0$ and $f(3) = 22 > 0$ implies there is a root between 1 and 3. Therefore, f(x) has two positive real roots and two complex roots. Now, to factor f(x), consider

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

$(x^2 - x + 1)^2 = x^4 + x^2 + 1 - 2x^3 + 2x^2 - 2x$

$(x^2 - x + 1)^2 = x^4 - 2x^3 + 3x^2 - 2x + 1$

$(x^2 - x + 1)^2 - 3x^2 = x^4 - 2x^3 - 2x + 1$

$(x^2 - x + 1)^2 - 3x^2 = f(x)$

$(x^2 - x + 1)^2 - 3x^2 = 0$ The left side of the equation may be factored as the difference of two squares.

$\left((x^2 - x + 1) - \sqrt{3}x\right)\left((x^2 - x + 1) + \sqrt{3}x\right) = 0$

$\left(x^2 - \left(1 + \sqrt{3}\right)x + 1\right)\left(x^2 - \left(1 - \sqrt{3}\right)x + 1\right) = 0$ Next, set each factor equal to zero and solve using the quadratic formula.

$x^2 - \left(1 + \sqrt{3}\right)x + 1 = 0$ $\qquad$ $x^2 - \left(1 - \sqrt{3}\right)x + 1 = 0$

$x = \dfrac{1 + \sqrt{3} \pm \sqrt{2\sqrt{3}}}{2}$ $\qquad$ $x = \dfrac{1 - \sqrt{3} \pm \sqrt{-2\sqrt{3}}}{2}$